Joint Entrance Screening Test 2021

The Institute of Mathematical Sciences, Chennai   11 April 2021

Subject: Theoretical Computer Science   10am to 1pm

Full Name in BLOCK LETTERS:

Registration Number :

C

Invigilator's Signature:

Instructions

- The paper is given in two parts: of these, Part A contains 25 multiple-choice questions.

  Unless your performance in Part A is satisfactory, your answers to questions in Part B will not be evaluated.

- Answers for Part A must be entered on the computer-readable answer sheet that is provided separately. Enter at most one choice for each question. Each correct answer carries 3 marks, each incorrect answer carries -1 mark. Unattempted questions fetch 0 marks.

- **The answer sheet for Part A will be collected by the invigilator at the end of seventy five (75) minutes.**

- Part B contains 09 questions and carries 48 marks. **You are expected to solve three of the eight-mark Questions (1, 2, 3, 4, 5) and two of the twelve-mark Questions (6, 7, 8, 9). However, you can attempt more questions if you wish. In that case, marks will be given to the best three answers for the eight-mark questions, and the best two answers for the twelve-mark questions.**

Give brief answers within the space provided. Do not use additional sheets for answers. **Answer to a new question must start on a fresh page. Write the question number in the space provided at the top of the page.** When an algorithm is asked for, give the required ideas, do not write a detailed program in a real programming language.

PART A

25 Multiple-choice Questions

The answer sheet for Part A will be collected by the invigilator at the end of seventy five (75) minutes.

1. Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function of arity n. It is said to **depend on** its ith argument if there exist $a_1, a_2, \ldots, a_n \in \{0,1\}$ such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, 1 - a_i, a_{i+1}, \ldots, a_n).$$

The number of Boolean functions of arity 2 that depend on both arguments is:

A) 7

B) 10

C) 12

D) 16

2. Let $x_1, x_2, \ldots, x_n$ be variables that take values true or false. A literal is a variable x_i or its negation $\neg x_i$. A clause is a disjunction of literals, for instance $(x_1 \lor \neg x_5 \lor x_9)$. We are given m clauses, each with exactly 3 distinct literals. No clause contains both a variable and its negation. We pick an assignment α to the n variables uniformly at random. Let K be the number of clauses satisfied by α. Which of the following statements is true? The expected value of K is:

A) $7m/8$

B) $mn/8$

C) $n/3$

D) Cannot be determined without knowing the clauses.

3. The sets S_N and T_N are defined as follows:

$$S_N = \left\{ i \in \{1,\dots,N\} \mid \left\lfloor \frac{i-1}{2}\right\rfloor < \left\lfloor \frac{i}{2}\right\rfloor \right\}, \quad T_N = \left\{ i \in \{1,\dots,N\} \mid \left\lceil \frac{i-1}{3}\right\rceil < \left\lceil \frac{i}{3}\right\rceil \right\}.$$

Which of these numbers is closest to the size of the set $S_N \cup T_N$?

A) $N/6$

B) $N/3$

C) $2N/3$

D) $5N/6$

4. A number X is picked from the set $\{1, 2, \ldots, 99\}$, according to an unknown random distribution. What is known about the distribution is that $\Pr[X = i] = \Pr[i + X = 100]$, for each $i \leq 50$. Which of the following is the value of $\sum_{i=1}^{99} \Pr[X \geq i]$?

A) 2

B) 50

C) 75

D) Cannot be determined from this data.

5. Which of the following statement is TRUE?

A) Regular languages are closed under countable union.

B) Regular languages are closed under union, finite or infinite.

C) Regular languages are closed under infinite union but not infinite intersection.

D) Regular languages are closed under finite union but not infinite union.

6. Which of the following is FALSE? A deterministic finite state automata cannot have

A) Multiple transitions on the same letter of the alphabet from the same state.

B) Multiple transitions between any two states on the same letter of the alphabet.

C) Multiple transitions on the same letter of the alphabet into the same state.

D) Self-loops on a state as well as transitions to other states on the same letter of the alphabet.

7. A language over $\Sigma = \{0,1\}$ is a set of finite length binary strings over Σ. The class of all infinite languages over Σ is

A) uncountably infinite

B) finite

C) countably infinite

D) empty

8. Given a 4-regular graph on n vertices, the number of 3-regular induced subgraphs on 9 vertices is

A) 0

B) $\binom{n}{9}$

C) $n(n-1)\ldots(n-8)$

D) n^9

9. An automorphism of a graph $G = (V, E)$ is a bijective mapping $\phi : V \to V$ preserving adjacencies, i.e., for all $u, v \in V$, $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E$. The number of automorphisms of a cycle on $n \geq 3$ vertices is

A) $n + 1$

B) $2n$

C) $n!$

D) none of the choices

10. In how many ways can you distribute two identical red balls and two identical blue balls into one white bin and one black bin?

A) 5

B) 9

C) 16

D) 64

11. An **oriented simple graph** is a simple undirected graph whose edges are directed from one vertex to another. The number of oriented simple graphs with four vertices and two edges is

A) 40

B) 50

C) 60

D) 70

12. A full binary tree is a rooted tree where every node has either zero or two children. The height of a leaf in such a tree is the number of edges in the path from the root to that leaf, and the height of the tree is the maximum height amongst all the leaves in the tree. Which of the following statements is TRUE?

A) There exists a full binary tree with 31 leaves, each leaf of height 5.

B) There exists a rooted tree of height 3 where every vertex has at most 3 children and there are 41 total vertices.

C) There exists a full binary tree with 11 vertices and height 6.

D) There exists a binary tree with 2 leaves and height 100.

13. Let $E = \{2, \{4, 5\}, 4\}$. Consider the statements: (a) $5 \in E$, (b) $\{5\} \in E$ and (c) $\{5\} \subset E$. Which of these statements is incorrect?

A) All of them

B) Only (a)

C) Both (b) and (c)

D) None of them

14. You are given two bowls of fruits, one containing five green apples and another containing five red apples. Of the green apples, three are contaminated with some harmful chemicals and the other two are normal. Of the red apples, two are contaminated and the other three are normal. We assume that the poison is rather weak and that it will require you to eat about two apples to make you seriously sick. You must choose one of the following two options

    (a) Eat two green apples (2 out of 5 are normal)

    (b) Eat three red apples (3 out of 5 are normal)

    Which of the following statements is TRUE?

    A) Option (a) gives you a strictly better chance of being healthy than option (b).

    B) Option (b) gives you a strictly better chance of being healthy than option (a).

    C) Both options give you the same chance of remaining healthy.

    D) None of the above.

15. An equilateral triangle, a square and a regular hexagon are inscribed in a circle. Which of the following is TRUE?

A) The hexagon has the largest area and the largest perimeter.

B) The hexagon has larger area but smaller perimeter than the triangle.

C) The hexagon has smaller area but larger perimeter than the square.

D) The hexagon has smallest area but largest perimeter.

16. Let A be the vertex-edge incidence matrix of a bipartite graph $G(X \cup Y, E)$, i.e., $a_{v,e} = 1$ if and only if the vertex $v \in X \cup Y$ is incident on the edge $e \in E$. Which of the following statements is TRUE?

A) The rank of A is always $= |X| + |Y|$.

B) The rank of A is $> |X| + |Y|$ for some graphs.

C) The rank of A is always $< |X| + |Y|$.

D) The rank of A is $= |X| + |Y|$ for some graphs and $< |X| + |Y|$ for some other graphs.

17. The order of a permutation π is the least n such that π^n, i.e., π composed with itself n times, is the identity permutation. Let $\ell_1, \ldots, \ell_k$ be the lengths of the cycles in the cycle-decomposition of a permutation π. Then the order of π is equal to

    A) The gcd of $\ell_1, \ldots, \ell_k$.

    B) The lcm of $\ell_1, \ldots, \ell_k$.

    C) The maximum of $\ell_1, \ldots, \ell_k$.

    D) The minimum of $\ell_1, \ldots, \ell_k$.

18. The statement $f(n) = n^{2(1+o(1))}$ is *equivalent* to which of the following?

A) there exists $\epsilon > 0$ and n such that $f(n)/n^2 \in [n^{-2\epsilon}, n^{2\epsilon}]$.

B) there exists $\epsilon > 0$ such that for n sufficiently large $f(n)/n^2 \in [n^{-2\epsilon}, n^{2\epsilon}]$.

C) for all $\epsilon > 0$ there exists n such that $f(n)/n^2 \in [n^{-2\epsilon}, n^{2\epsilon}]$.

D) for all $\epsilon > 0$ and n sufficiently large $f(n)/n^2 \in [n^{-2\epsilon}, n^{2\epsilon}]$.

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19. Which of following is the correct statement for the recurrence $T_n = 2T_{n-1} + T_{n-2}$, where $T_0 = 0$, $T_1 = 1$ and F_n is the nth Fibonacci number?

A) $T_n = O(2^n)$

B) $T_n \leq 2F_n$, for $n > 2$

C) $T_n > 2F_n$, for $n > 2$

D) $T_n = \Theta(F_n)$

20. Suppose Euler wrote in a letter to Johann Bernoulli that the polynomial $x^2 - x + 41$ evaluates to a prime for all natural numbers. If you were Johann Bernoulli which of the following would be the correct response?

A) Agree

B) Disagree

C) Can't say

D) Probably true

21. For a prime $p > 2$, and all natural numbers $n > 1$, which of the following polynomials is congruent modulo p to the polynomial $(n + x)^p$?

A) $(1 + x)$

B) $(n + x^2)$

C) $(np + x^p)$

D) $(n + x^p)$

22. Suppose a graph $G = (V, E)$ satisfies $|E| < 3|V| - 1729$. Then the average degree of the vertices in G cannot be

   A) 6

   B) 5

   C) 4

   D) 3

23. The number of pairs (x, y) where $1 \leq x, y \leq 100$ and x/y is a natural number that is a perfect square

    is

    A) 117

    B) 125

    C) 143

    D) 153

24. Let A be a square matrix of dimension strictly greater than one. The characteristic polynomial of A is defined as $p_A(x) = \det(xI - A)$. Consider the following argument in proof of the claim $p_A(A) = 0$:

Substituting $x = A$ in the definition of the characteristic polynomial we obtain $p_A(A) = \det(A - A) = 0$.

Which of the following statements is TRUE?

A) The argument is correct.

B) The argument is fallacious.

C) The argument is correct for a class of matrices.

D) None of the above.

25. Let $m < n$. The number of arithmetic operations, in the *worst case*, that are required to reduce an $m \times n$ matrix with real numbers to row reduced echelon form using Gaussian elimination is

A) $\Theta(mn)$

B) $\Theta(m^2 n)$

C) $\Theta(mn^2)$

D) $\Theta(m^2 n^2)$

Recall that the best three answers out of Questions 1 to 5, and the best two answers out of questions 6 to 9 will be chosen for credit.

1. The median of a sorted array of n distinct integers is either the value at the index $\lceil n/2 \rceil$, if n is odd, or the average of the values at the indices $(n/2)$ and $(n/2)+1$, if n is even. Given two sorted arrays of n distinct integers each, give an $O(\log n)$ algorithm to find the median of the union of the two arrays. Note that the union is an array of $2n$ elements, not necessarily distinct. [8 Marks]

2. Let $w \in \Sigma^*$ be a word. Construct a deterministic finite state automaton M_w on the alphabet Σ such that the language accepted by M_w is exactly the set of *subwords* of w. If $w = a_1 a_2 \ldots a_n$ then $a_{i_1} a_{i_2} \ldots a_{i_m}$ is a subword of w, where $m \le n$, $i_1 < i_2 < \cdots < i_m$. [8 Marks]

3. Boolean formulas are built from boolean variables using connectives, such as, negation ("not"), conjunction ("and") and disjunction ("or"). We say that a boolean formula α is *valid* if it evaluates to true for all assignments of true/false to each of the boolean variables in α. The boolean formula "$\alpha \equiv \beta$" is read as: α holds *if and only if* β holds, that is, a true/false assignment to each of the variables in α and β either makes both of them true or both of them false. Let α be a formula which has only boolean variables, equivalence ($\equiv$), and no other connectives; e.g., $(x \equiv x) \equiv (y \equiv y)$. Show that α is *valid* if and only if every boolean variable occurring in α occurs an *even* number of times. [8 Marks]

4. Let L_1, L_2 be regular languages such that $L_1 \subseteq L_2$ and $L_2 \setminus L_1$ is infinite. Show that there is a regular language L such that $L_1 \subseteq L \subseteq L_2$ and both $L \setminus L_1$ and $L_2 \setminus L$ are infinite. [8 Marks]

5. Suppose that $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, where $n \geq 2$, and every pair of these intervals has a non-empty intersection, that is, $I_i \cap I_j \neq \emptyset$ for all, $1 \leq i, j \leq n$. Prove that the intersection of all these intervals is nonempty, that is, $I_1 \cap I_2 \cap \ldots \cap I_n \neq \emptyset$. [8 Marks]

6. A vertex cover S in a graph $G(V, E)$ is a subset of V such that for every edge $(u, v) \in E$, at least one of u or v is in S.

   (a) Give a polynomial time algorithm to find a minimum-size vertex cover in a tree.

   (b) Suppose $G(V, E)$ is a graph with a subset $S \subset V$ of size k such that $G \setminus S$ is a forest. Give a $O(2^k n^{O(1)})$ algorithm to find a minimum-size vertex cover in G. Assume that S is given as a part of the input.

[12 Marks]

7. Let P be a set of n points distributed uniformly on the unit circle, that is, the lengths of the arcs between all pairs of adjacent points are the same. Let S be a set of m line segments with endpoints in P. The endpoints of the m segments are not necessarily distinct. Two segments are said to be disjoint if they do not intersect, even at their endpoints.

   (a) Derive a recurrence for $D(m, n)$, the size of a largest subset of segments in S such that every pair in S is disjoint.

   (b) Describe an algorithm to compute $D(m, n)$. For full credit, the algorithm should run in $O(mn)$ steps.

[12 Marks]

8. Prove any two of the following claims:

   (a) Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

   (b) Let G be a graph such that for all $u, v \in V(G)$, $u \neq v$, $|N(u) \cap N(v)|$ is odd, where $N(u) \subseteq V$ denotes the neighbourhood of a vertex u. Then show that the number of vertices in G is odd.

   (c) If G is a connected graph then show that any two paths of maximum length have a vertex in common.

[12 Marks]

9. Suppose you are given an array $A[1 \ldots n]$ of integers, each of which may be positive, negative, or zero. A contiguous subarray $A[i \ldots j]$ is called a *positive interval* if the sum of its entries is greater than zero. Describe and analyse an algorithm to compute the minimum number of positive intervals that cover every positive entry in A. For example, given the following array as input, the algorithm should output 3. If every entry in the input array is negative, the algorithm should output 0.

$$A = (\underbrace{+3, -5, +7, -4, +1}_{\text{sum}=2}, -8, \underbrace{+3, -7, +5}_{\text{sum}=1}, -9, \underbrace{+5, -2, +4}_{\text{sum}=7})$$

[12 Marks]

Answer to Question Number:

Answer to Question Number:

Pages for answer to questions numbered 1 to 5

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Answer to Question Number:

Pages for answer to questions numbered 6 **to** 9

Answer to Question Number:

Pages for answer to questions numbered 6 **to** 9

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